Robustness to strategic uncertainty in price competition

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Abstract. We model a player's uncertainty about other players' strategy choices as probability distributions over their strategy sets. We call a strategy profile robust to strategic uncertainty if it is the limit, as uncertainty vanishes, of some sequence of strategy profiles in each of which every player's strategy is optimal under under his or her uncertainty about the others. We apply this definition to Bertrand games with a continuum of equilibrium prices and show that our robustness criterion selects a unique Nash equilibrium price. This selection agrees with recent experimental findings.

Keyword: Nash equilibrium, refinement, strategic uncertainty, price competition
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1. Introduction

Price competition is usually modelled as a game with a continuum of prices available to each competitor. If the good is homogeneous, payoff discontinuities naturally arise. For instance, in canonical Bertrand competition, the slightest undercutting of competitors' lowest price results in a discrete upward jump in sales. As is well-known, if the competing firms have the same constant average cost, then their common and constant marginal cost is the unique Nash equilibrium market price. By contrast, if marginal costs are strictly increasing, then there is a whole continuum of equilibrium market prices (Dastidar, 1995). In both settings, even the slightest uncertainty about competitors' price choices might lead firms to deviate from any given equilibrium price vector. It is then arguably reasonable to require equilibria to be robust to small amounts of uncertainty about other players' strategies.

In this note we formalize a notion of strategic uncertainty and propose a criterion for robustness to such uncertainty. Our approach is roughly as follows. Given any game with finitely many players in which each player's strategy set is a continuum, a player's uncertainty about others' strategy choices is represented by a probability distribution anchored at those strategies and scaled with a parameter $t \geq 0$. The probability distributions are assumed to be atomless and have standard properties. For each level of this perturbation parameter t, we define a t-equilibrium as a Nash equilibrium of the accordingly perturbed game, in which each player strives to maximize her expected payoff under her strategic uncertainty. We call a strategy profile of the original game robust to strategic uncertainty if there exists a collection of probability distributions, one for each player, such that some accompanying sequence of t-equilibria converges to this profile, as the perturbation parameter t tends to zero. We call the strategy profile strictly robust if this holds for all probability distributions in the admitted class. The aim of this note is limited: we here only study in detail the implications of these definitions for a particular class of games.

We apply this refinement to Bertrand competition.[1] By way of a simple duopoly example with constant and identical marginal costs, we first show that our refinement admits the unique and weakly dominated Nash equilibrium. Nevertheless, when marginal costs are strictly increasing, our robustness criterion selects a unique strategy profile out of the continuum of Nash equilibria. This prediction agrees with recent findings in experimental studies of (discretized versions of) Dastidar's (1995) model, see Abbink and Brandts (2008) and Argenton and Müller (2009).[2] Abbink

[1] Following Vives (1999, p.117), we take Bertrand competition to mean that (a) sellers simultaneously choose their prices and (b) each firm has to serve all its clients at the price it has chosen.

[2] Abbink and Brandts (2008) ran experiments with fixed groups of two, three, and four identical firms. They find that duopolists are often able to collude on the joint profit-maximizing price. However, the lowest price in the range of Nash equilibria which involves no loss in case of miscoordination (24 in their specification), a much smaller number than the collusive price, is also an

and Brandts (2008) remark that "[that] price level (...) is not predicted by any benchmark theory [they] are aware of" (p. 3). The present refinement provides a theoretical foundation for their finding. Heuristically, strategic uncertainty, as modelled here, results in perturbed profit functions that are continuous, since the likelihood of serving the entire market is continuous in one's own price. The deviation incentives are asymmetric, though. For high Nash equilibrium prices, a strategically uncertain player has an incentive to cut her price, since she has a lot to lose if others' prices lie a bit below her price and little to gain if they lie a bit above it. Conversely, for low Nash equilibrium prices, each player has an incentive to raise her price, since she has a lot to loose if others' prices lie a bit above her price and little to gain if they lie a bit below. The only price that is robust to strategic uncertainty is the price at which the monopoly profit is zero. This is also the maximal Nash equilibrium price in the limit as the number of competitors tend to infinity. At that price, and no other price, the incentives to move up and down for a strategically uncertain player are of the same order of magnitude.

Our robustness criterion is closely related to Selten's (1975) "substitute perfection". Selten defined a Nash equilibrium in a finite game to have this property if there exists a sequence of completely mixed strategy profiles, converging to the equilibrium in question, such that each player's equilibrium strategy is a best reply to all but finitely many strategy profiles in the sequence. Substitute perfect equilibria exist in all finite games, and, as Selten (1975) shows, they coincide with (trembling-hand) perfect equilibria. However, in generic non-linear games with continuum strategy spaces, no Nash equilibrium is literally substitute perfect, the reason being that small perturbations of players' beliefs induce small changes in their best replies (while the discreteness in finite games allows best replies to remain unchanged under such perturbations).

Simon and Stinchcombe (1995) extended Selten's perfection criterion to games with compact strategy sets and continuous payoff functions. By contrast, we here focus on a class of games with discontinuous payoff functions. Binmore (1987) and Carlsson (1991) study equilibrium selection in the Nash demand game (Nash, 1953), which also admits a continuum of equilibria. Both authors assume that players "tremble." By contrast, players do not "tremble" in our model; they are only uncertain about other players' action. Carlsson and Ganslandt (1998) investigate "noisy equilibrium selection" in symmetric coordination games and derive results that agree with the experimental findings on minimal effort games in Van Huyck et al. (1990). While Carlsson and Ganslandt's (1998) study is tailored to the minimal effort game, we

attractor of play. With more than two firms in a market, it actually is the predominant market price. This outcome is also observed in the complete information, symmetric treatment in Argenton and Müller (2009).

here make general assumptions concerning players' beliefs, assumptions that permit an operational definition of robustness to strategic uncertainty for a large class of games. Our approach is related to that in Friedman and Mezzetti (2005), who introduce a notion of "robust equilibria" in finite games, as the limit of sequences of "random belief equilibria." In a random belief equilibrium, all players' beliefs are random variables, and a player's best-reply distribution, implied by her belief distribution, is required to be consistent, in terms of statistical expectation, with others' beliefs about that player's action. By contrast, we analyze continuum-action games and impose no such consistency requirement.

The application is here to Bertrand games in which firms are committed to serve any demand addressed to them at the posted price; they cannot turn customers down or ration them. As mentioned by Vives (1999), for certain utilities and auctions, provision is legally mandated, and in other markets firms have a strong incentive to serve all their clients, especially in industries in which customers have an on-going relationship with suppliers (subscription, repeat purchases, etc.) or where the costs of restricting output in real time are high. There are a number of papers focused on price competition with convex costs. Dixon (1990) studies such competition when firms are not obliged to serve all demand, but incur a cost when turning costumers down. He shows that under such circumstances there may still exist a continuum of pure-strategy Nash equilibria. Spulber (1995) assumes that firms are uncertain about rivals' costs and shows that there exists a unique symmetric Nash equilibrium in pure strategies. As the number of firms grows, equilibrium pricing strategies tend to average cost pricing. Chowdhury and Sengupta (2004) show that, in Bertrand games with convex costs, there exists a unique coalition-proof Nash equilibrium (in the sense of Bernheim, Peleg and Whinston 1987), which converges to the competitive outcome under free entry. Our criterion selects another price, which, moreover, does not depend on the number of firms.

2. Robustness to strategic uncertainty

Let $G = (N, S, \pi)$ be an n-player normal-form game with player set $N = \{1, ..., n\}$, in which the pure-strategy set of each player is the real line, $S_i = \mathbb{R}$, and thus $S = \mathbb{R}^n$ is the set of pure-strategy profiles $\mathbf{s} = (s_1, ..., s_n)$, and $\pi : S \to \mathbb{R}^n$ is the combined payoff-function, with $\pi_i(\mathbf{s})$ being the payoff to player i when $\mathbf{s}$ is played.[3]

Let $\mathcal{F}$ be the class of cumulative probability distribution functions $F : \mathbb{R} \to [0, 1]$ with everywhere positive and continuous density $f = F'$ and with non-decreasing hazard rate, that is, such that the *hazard-rate function* $h : \mathbb{R} \to \mathbb{R}_+$, defined by

$$h(x) = \frac{f(x)}{1 - F(x)},$$

[3]See below for how this machinery can be adapted to restrictions on strategy sets.

is non-decreasing.[4] Examples of probability distributions with this property are the normal, exponential and Gumbel distributions. Bagnoli and Bergstrom (2005) show that a sufficient condition for this property is that f be *log-concave*, that is, that $\log f$ be a (strictly) concave function.[5]

Definition 1. *For any given $t \geq 0$, a strategy profile $\mathbf{s}$ is a **t-equilibrium** of G if, for each player i, the strategy s_i maximizes i's expected payoff under the probabilistic belief that all other players' strategies are random variables of the form*

$$\tilde{s}_{ij} = s_j + t \cdot \varepsilon_{ij} \tag{1}$$

for some statistically independent "noise" terms $\varepsilon_{ij} \sim \Phi_{ij}$, where $\Phi_{ij} \in \mathcal{F}$ for all $j \neq i$.

Remark 1. *For $t = 0$, this definition coincides with that of Nash equilibrium.*

Remark 2. *For $t > 0$, the random variable $\tilde{s}_{ij}$ has the c.d.f. $F_{ij}^t \in \mathcal{F}$ defined by*

$$F_{ij}^t(x) = \Phi_{ij}\left(\frac{x - s_j}{t}\right) \quad \forall x \in \mathbb{R}.$$

Note that we do not require that noise terms are symmetric or have expectation zero, only that Φ_{ij} has a non-decreasing hazard rate. In particular, in a t-equilibrium players may believe that others have a systematic tendency to deviate upwards or downwards.

Example 1. *Let Φ_{ij} be a normal distribution, $N(\mu, \sigma)$, with $\mu = \sigma = 1$, and hence $E[\tilde{s}_{ij}] = s_j + t$. Then the density f_{ij}^t is skewed to the right, as shown in the diagram below for $s_j = 10$, and $t = 0.3$ (thick), $t = 0.1$ (dashed) and $t = 0.05$ (thin).*

Let $\tilde{s}_{-i}$ denote the $(n-1)$-vector of random variables $(\tilde{s}_{ij})_{j \neq i}$. We note that a t-equilibrium is a Nash equilibrium of a game with perturbed payoff functions:

Remark 3. *Let $t > 0$ and $\Phi_{ij} \in \mathcal{F}$ for all $i \in N$ and $j \neq i$. A strategy profile $\mathbf{s} \in S$ is a t-equilibrium of $G = (N, S, \pi)$, with $\varepsilon_{ij} \sim \Phi_{ij}$, if and only if it is a Nash*

[4] For the present application, this condition is sufficient. However, in other applications the methodology developed here may require, for asymmetric noise distributions, that a corresponding condition be imposed on the "reversed hazard rate", the density divided by the mass of the left tail.

[5] The log-concavity assumption is common in the economics literature and has applications in mechanism design, game theory and labor economics, see Bagnoli and Bergstrom (2005).

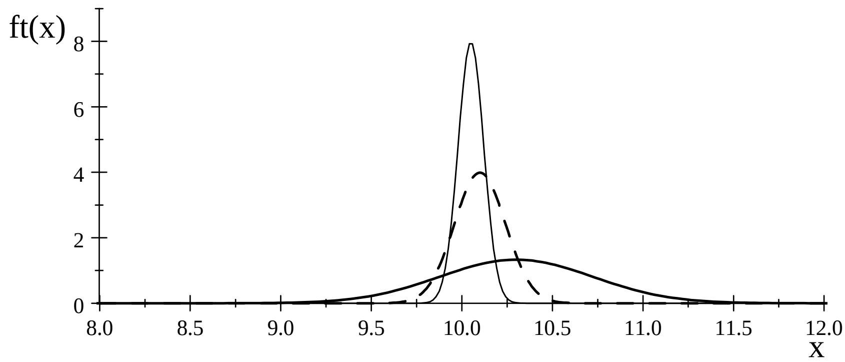

Figure 1: Player i's probabilitic belief about s_j

equilibrium of the game $G^t = (N, S, \pi^t)$*, where*

$$\begin{aligned}
\pi_i^t(\mathbf{s}) &= \mathbb{E}\left[\pi_i\left(s_i, \tilde{s}_{-i}\right)\right] \\
&= \int_{S_1} .. \int_{S_{i-1}} \int_{S_{i+1}} .. \int_{S_n} \pi_i\left(s_i, s_{-i}\right) dF_{i1}^t\left(s_1\right) .. dF_{i,i-1}^t\left(s_{i-1}\right) dF_{i,i+1}^t\left(s_{i+1}\right) .. dF_{in}^t\left(s_n\right) \\
&= \frac{1}{t^{n-1}} \int .. \int .. \int \left[\prod_{j \neq i} \phi_{ij}\left(\frac{x_j - s_j}{t}\right) \pi_i\left(s_i, x_{-i}\right)\right] dx_1 .. dx_{i-1} dx_{i+1} .. dx_n
\end{aligned}$$

We are now in a position to define robustness to strategic uncertainty.

Definition 2. *A strategy profile* $\mathbf{s}^*$ *in the game* G *is* ***robust to strategic uncertainty*** *if there exists a collection of c.d.f:s* $\{\Phi_{ij} \in \mathcal{F} : \forall i \in N, j \neq i\}$ *and an accompanying sequence of* t*-equilibria,* $\langle \mathbf{s}^{t_k} \rangle_{k \in \mathbb{N}}$ *with* $t_k \downarrow 0$*, such that* $s^{t_k} \to s^*$ *as* $k \to +\infty$*. The strategy profile* $\mathbf{s}^*$ *is* ***strictly*** *robust to strategic uncertainty if this holds for all collections of c.d.f:s* $\{\Phi_{ij} \in \mathcal{F} : \forall i \in N, j \neq i\}$*.*

Remark 4. *This definition can be adapted as follows to games in which the strategy set of each player* j *is an interval* $S_j = [0, b_j]$ *for some* $b_j > 0$*. For any* $\Phi_{ij} \in \mathcal{F}$*, let*

$$F_{ij}^t(x) = \frac{\Phi_{ij}\left(\frac{x - s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}{\Phi_{ij}\left(\frac{b_j - s_j}{t}\right) - \Phi_{ij}\left(-\frac{s_j}{t}\right)}$$

This defines a c.d.f. for $\tilde{s}_{ij}$ *with support* $[0, b_j]$*, such that, for any* $s_j, x \in [0, b_j]$*:*

$$\lim_{t \to 0} F_{ij}^t(x) = \begin{cases} 0 & \text{if } x < s_j \\ 1 & \text{if } x \geq s_j \end{cases}$$

Taking expectations with respect to such c.d.f:s F_{ij}^t, one obtains a perturbed game with payoff functions

$$\begin{aligned}\pi_i^t(\mathbf{s}) &= \mathbb{E}\left[\pi_i(s_i,\tilde{s}_{-i})\right] \\ &= \frac{1}{t^{n-1}}\int\cdots\int\cdots\int\left[\prod_{j\neq i}\phi_{ij}^t\left(\frac{x_j-s_j}{t}\right)\pi_i(s_i,x_{-i})\right]dx_1...dx_{i-1}dx_{i+1}...dx_n\end{aligned}$$

where

$$\phi_{ij}^t\left(\frac{x_j-s_j}{t}\right)=\frac{\phi_{ij}\left(\frac{x-s_j}{t}\right)}{\Phi_{ij}\left(\frac{b_j-s_j}{t}\right)-\Phi_{ij}\left(-\frac{s_j}{t}\right)}. \tag{2}$$

We note that for any interior strategy profile, $\mathbf{s}\in\times_{i\in N}(0,b_i)$, our robustness criterion is the same, whether or not the noise terms are fitted to the strategy sets in this way: for any $s_j\in(0,b_j)$, the denominator in (2) converges to 1 and its derivative converges to zero. If instead $S_i=\mathbb{R}_+$ for all players i, then all properties are retained by setting

$$F_{ij}^t(x)=\frac{\Phi_{ij}\left(\frac{x-s_j}{t}\right)-\Phi_{ij}\left(-\frac{s_j}{t}\right)}{1-\Phi_{ij}\left(-\frac{s_j}{t}\right)}. \tag{3}$$

As we show in the subsequent sections, this definition of robustness selects a unique Nash equilibrium out of a continuum of equilibria in a class of price competition games. Before embarking on that analysis, let us briefly consider the canonical Bertrand model of pure price competition.

Example 2. *Consider two identical firms, each with constant unit cost $c>0$, in a simultaneous-move pricing game à la Bertrand in a market for a homogeneous good. Let the demand function be linear, $D(p)=a-p$, for all $p\in[0,a]$ with $a>c$.*[6] *Then, the monopoly profit function, $\Pi(p)=(a-p)(p-c)$, is strictly concave with a unique maximum at $p^m=(a+c)/2$ and $\Pi(p^m)>0$. By contrast, the unique duopoly Nash equilibrium, $p_1=p_2=c$, results in zero profits. This Nash equilibrium is weakly dominated. Nevertheless, it is robust to strategic uncertainty. For sufficiently small degrees of strategic uncertainty, both firms will set their prices a little bit above marginal cost, and less so, the less uncertain they are. To see this, suppose that $\varepsilon_{ij}\sim\Phi\in\mathcal{F}$.*[7] *For each $t>0$ and all $p_1,p_2\in[0,a]$,*

$$\pi_i^t(p_i,p_j)=\left[1-\frac{\Phi\left(\frac{p_i-p_j}{t}\right)-\Phi\left(-\frac{p_j}{t}\right)}{1-\Phi\left(-\frac{p_j}{t}\right)}\right]\cdot\Pi(p_i)\qquad i=1,2,\ j\neq i.$$

[6] To keep the intuition clear, we take a simple functional form but the argument extends to general demand curves.

[7] We focus on symmetric error distributions in this example only for expositional convenience. The Nash equilibrium is robust to strategic uncertainty under asymmetric distributions as well.

This can be rewritten as

$$\pi_i^t(p_i, p_j) = [1 - \Phi(-p_j/t)]^{-1} \cdot \left[1 - \Phi\left(\frac{p_i - p_j}{t}\right)\right] \cdot \Pi(p_i) \qquad i = 1, 2,\ j \neq i,$$

where the first factor is positive and independent of p_i. *A necessary first-order condition for symmetric* t*-equilibrium*[8] *is thus that*

$$t \cdot \frac{\Pi'(p_i)}{\Pi(p_i)} = \frac{\phi(0)}{[1 - \Phi(0)]} \qquad i = 1, 2,\ j \neq i. \tag{4}$$

The RHS of (4) is a positive constant. Consequently, in the perturbed game, it is never optimal to choose $p_i \leq c$ *or* $p_i \geq p^m$. *Hence, without loss of generality, we restrict attention to* $p_i \in (c, p^m)$. *On this interval, the LHS is a continuous and strictly decreasing function that runs from plus infinity to zero. Hence, there exists a unique symmetric* t*-equilibrium price,* p^t*, for every* $t > 0$. *Moreover, as* $t \to 0$*, the denominator of the LHS has to tend to zero for (4) to hold. Consequently,* $p^t \downarrow c$. *The diagram below shows how the* t*-equilibrium price* p^t *depends on* t*, when* Φ *is the standard normal distribution,* $a = 1$ *and* $c = 0.2$.

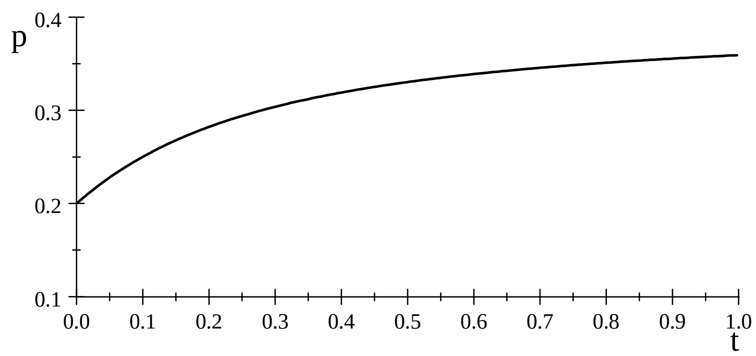

Figure 2: The $t-$equilibrium price as a function of t in the standard Bertrand game.

3. PRICE COMPETITION WITH CONVEX COSTS

There are $n \geq 2$ firms $i \in N = \{1, 2, ..., n\}$ in a market for a homogeneous good. Aggregate demand $D : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$ is twice differentiable and such that $D(0) = q^{\max} \in \mathbb{R}$ and $D(p^{\max}) = 0$ for some $p^{\max}, q^{\max} > 0$.[9] Moreover, we assume that

[8] It is easily verified that there does not exist any asymmetric t-equilibrium.

[9] In this section, we follow closely Dastidar (1995).

$D'(p) < 0$ for all $p \in (0, p^{\max})$. All firms i simultaneously set their prices $p_i \in \mathbb{R}_+$. Let $\mathbf{p} = (p_1, p_2, ..., p_n)$ be the resulting strategy profile (or *price vector*). The minimal price, $p_0 := \min \{p_1, p_2, ..., p_n\}$, will be called *the (going) market price*. Let k be the number of firms that quote the going market price, $k := | \{i : p_i = p_0\} |$. Each firm i faces the demand

$$D_i(\mathbf{p}) := \begin{cases} D(p_0)/k & \text{if } p_i = p_0 \\ 0 & \text{otherwise} \end{cases}$$

All firms have the same cost function, $C : \mathbb{R}_+ \longrightarrow \mathbb{R}_+$, which is twice differentiable with $C(0) = 0$ and $C', C'' > 0$. Each firm is required to serve all demand addressed to it at its posted price. The profit to each firm i is thus

$$\pi_i(\mathbf{p}) = \begin{cases} p_0 D(p_0)/k - C\left[D(p_0)/k\right] & \text{if } p_i = p_0 \\ 0 & \text{otherwise} \end{cases} \tag{5}$$

This defines a simultaneous-move n-player game G in which each player i has pure-strategy set $\mathbb{R}_+$ and payoff function $\pi_i : \mathbb{R}_+^n \to \mathbb{R}$, defined in equation (5). A strategy profile $\mathbf{p}$ will be called *symmetric* if $p_1 = ... = p_n$, and we will call a price $p \in \mathbb{R}_+$ a *symmetric Nash equilibrium price* if $\mathbf{p} = (p, p, ..., p)$ is a Nash equilibrium of G. For each positive integer $k \leq n$ and non-negative price p, let

$$v_k(p) = pD(p)/k - C\left[D(p)/k\right]$$

This defines a finite collection of twice differentiable functions, $\langle v_k \rangle_{k \in \{1,2,..,n\}}$, where $v_k(p)$ is the profit to each of k firms if they all quote the same price p and all other firms post higher prices (so that p is the going market price). In particular, v_1 defines the profit to a monopolist as a function of its price p.

We impose one more condition on C and D, namely, that the associated monopoly profit function, v_1, is strictly concave. More exactly, we assume that $v_1'' < 0$ and $v_1'(p^{mon}) = 0$ for some price $p^{mon} \in (0, p^{\max})$. Since the cost function is strictly convex by assumption, this concavity assumption on v_1 effectively requires the demand function to be "not too convex".[10] We have $v_1(p^{mon}) \geq 0$. By convexity of the cost function, there exists prices $p \in (0, p^{\max})$ at which all n firms, when quoting the same price p, make positive profits, $v_n(p) > 0$.

Game G has a continuum of symmetric Nash equilibria.[11] For any number of firms, $n \geq 2$, let $\check{p}_n \in (0, p^{\max})$ be the price p at which $v_n(p) = 0$ and let $\hat{p}_n \in (0, p^{\max})$ be the price p at which $v_n(p) = v_1(p)$. Dastidar (1995, Lemmas 1, 5 and 6) shows existence and uniqueness of $\check{p}_n$ and $\hat{p}_n$ and that $\check{p}_n < \hat{p}_n$. As also shown in Dastidar

[10] This is a more stringent assumption than the one made in Dastidar (1995), who instead assumes that there exists a unique monopoly price.

[11] Dastidar (1995) and Weibull (2006) have shown existence and multiplicity of Nash equilibria under weaker conditions.

(1995, Proposition 1), all prices in the interval $P_n^{NE} = [\check{p}_n, \hat{p}_n]$ are symmetric Nash equilibrium prices in the game G, and no price outside this interval is a symmetric Nash equilibrium price.

As shown in Dastidar (1995, Lemmas 4 and 6), there exists a unique price $\bar{p}$ at which a monopolist makes zero profit, $v_1(\bar{p}) = 0$, and, moreover, $\bar{p} \in (\check{p}_n, \hat{p}_n)$. Dastidar (1995, Lemma 7) also shows that both $\check{p}_n$ and $\hat{p}_n$ are strictly decreasing in n. In the present setting, it is easily verified that $\check{p}_n \downarrow 0$ and $\hat{p}_n \downarrow \bar{p}$, and hence $P_n^{NE} \to (0, \bar{p}]$, as $n \to \infty$.

Example 3. *Consider a duopoly with identical firms with quadratic cost functions, $C(q) = cq^2$, where $c = 0.2$, and linear aggregate demand: $D(p) = \max\{0, 1-p\}$. The diagram below shows the graphs of v_1 (dashed curve) and v_2 (solid curve). The associated set, P_2^{NE}, is the interval $[1/11, 3/13]$, indicated by the two solid vertical lines, and $\bar{p} = 1/6$ is indicated by the dashed vertical line.*

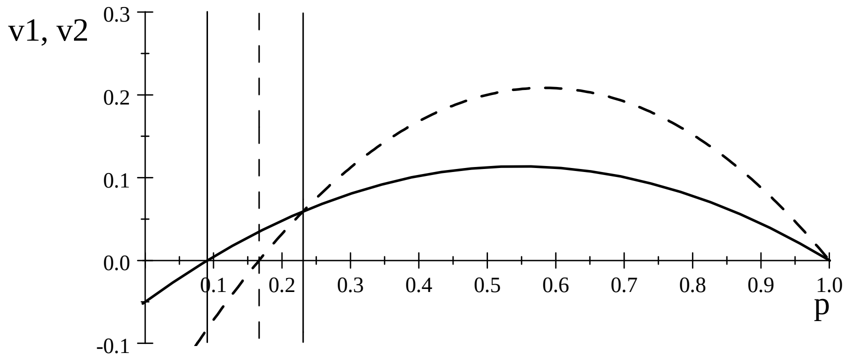

Figure 3: Monopoly (dashed) and duopoly (solid) profits, as functions of a common price p.

We make two further observations. First, that $\hat{p}_n$ cannot exceed the monopoly price, and second, that the pricing game G admits no asymmetric Nash equilibrium.

Proposition 1. $\hat{p}_n \leq p^{mon}$ *for all* $n > 1$.

Proof: Dastidar (1995; Lemma 3) shows that, if $v_n(p) \geq v_1(p)$ then $v_1(p) > v_1(p-\alpha)$, $\forall \alpha > 0$ for $p - \alpha \in [0, p)$. So, if p is a Nash equilibrium, then the left-derivative of v_1 at p must be positive. The concavity of v_1 implies that $\hat{p}_n \leq p^{mon}$. **End of proof**.

Proposition 2. *Every Nash equilibrium in G is symmetric.*

Proof: Let $(p_1, ..., p_n)$ be a Nash equilibrium. Suppose, first, that $p_i < \min_{j \neq i} p_j$ for some i. If $p_i < \hat{p}_n$, then firm i could increase its profit by unilaterally increasing its price. Hence, $p_i \geq \hat{p}_n$. If $p_i \leq p^{mon}$, then any firm $j \neq i$ could increase its profit by unilaterally decreasing its price to p_i and earn $v_2(p_i) > 0$ instead of zero. If $p_i > p^{mon}$ then firm i can increase its profit by a unilateral deviation to p^{mon}. Hence, $p_i \geq \min_{j \neq i} p_j$ for all i. Suppose, secondly, that $p_i = \min_{j \neq i} p_j$ and that $p_k > p_i$ for some k. Either $v_{|j \in N: p_j = p_i|}(p_i) > 0$ or $v_{|j \in N: p_j = p_i|}(p_i) = 0$. (If $v_{|j \in N: p_j = p_i|}(p_i) < 0$, then i can profitably deviate to $p^{\max}$ and earn zero profit.) In any case, k can profitably deviate to p_i and make a positive profit since by strict convexity of C, if $v_l(p) \geq 0$, then $v_{l+1}(p) > 0$. Hence, $p_i = p_j$ for all $i, j \in N$.**End of proof**.

4. ROBUST PRICE EQUILIBRIUM

We proceed to apply the robustness definition from Section 2 to the pricing game described in Section 3. Let $t > 0$ and suppose that a firm i holds a probabilistic belief of form (1) about other firms' prices. For any price p_i that firm i might contemplate to set, its subjective probability that any other firm will choose exactly the same price is zero. Hence, with probability one, its own price will either lie above the going market price or it will be the going market price and all other firms' prices will be higher, so i will then be a monopolist at its price p_i. From equation (3), each firm i's payoff function in the perturbed game $G^t = (N, S, \pi^t)$ is, for any $t > 0$, defined by

$$\pi_i^t(\mathbf{p}) = v_1(p_i) \cdot \left(\prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{-p_j}{t} \right) \right]^{-1} \right) \cdot \left(\prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{p_i - p_j}{t} \right) \right] \right) \tag{6}$$

The second factor being positive and independent of p_i, a price profile $\mathbf{p}$ is a Nash equilibrium of G^t if and only if

$$p_i \in \arg \max_{p \in [\bar{p}, p^{mon}]} u_i^t(p, \mathbf{p}_{-i}) \qquad \forall i, \tag{7}$$

where

$$u_i^t(\mathbf{p}) = v_1(p_i) \cdot \prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{p_i - p_j}{t} \right) \right]$$

and the restriction $p \in [\bar{p}, p^{mon}]$ is non-binding, since $v_1(p) < 0$ for all $p < \bar{p}$, $v_1(p) > 0$ for all $p \in (\bar{p}, p^{mon})$, and $v_1'(p) < 0$ for all $p > p^{mon}$. For any $t > 0$, let $\bar{G}^t$ be the normal-form game $(N, [\bar{p}, p^{mon}]^n, u^t)$. We have established

Lemma 1. *For any $t > 0$, a price profile $\mathbf{p}$ is a t-equilibrium in the pricing game G if and only if it is a Nash equilibrium of the game $\bar{G}^t$.*

Proposition 3. *Let $t > 0$ and assume that $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$. Then $\bar{G}^t$ has at least one Nash equilibrium. Moreover, any such Nash equilibrium $\mathbf{p}^t$ is interior.*

Proof: The strategy sets in $\bar{G}^t$ are compact and all payoff functions are continuous. Existence of Nash equilibrium follows if, moreover, each player's payoff is quasi-concave in the player's own strategy. To see whether this is the case, differentiate i's payoff with respect to p_i:

$$\frac{\partial u_i^t(\mathbf{p})}{\partial p_i} = \prod_{j \neq i} \left[1 - \Phi_{ij}\left(\frac{p_i - p_j}{t}\right)\right] \cdot \left[v_1'(p_i) - \frac{v_1(p_i)}{t} \sum_{j \neq i} h_{ij}\left(\frac{p_i - p_j}{t}\right)\right]$$

where h_{ij} is the hazard-rate function of Φ_{ij}. This is a continuous function of p_i. The expression in the first square bracket is positive and decreasing. Moreover, v_1' and v_1 are both positive on $(\bar{p}, p^{mon})$, with v_1' decreasing from a positive value towards zero, and v_1 increasing from zero to a positive value (by the assumed concavity of v_1). Since hazard-rates are non-decreasing by assumption, each term in the sum is non-decreasing in p_i. Hence, the expression in the second square bracket is decreasing from a positive to a negative value. Hence, $\partial u_i^t / \partial p_i$ is decreasing in p_i, so u_i^t is concave in p_i on $[\bar{p}, p^{mon}]$, and thus also quasi-concave. This establishes existence. Clearly, no price can lie on the boundary of the strategy set in $\bar{G}^t$, since $\partial u_i^t / \partial p_i$ is positive at its left boundary and negative at its right boundary. **End of proof.**

Theorem 1. *The Nash equilibrium $(\bar{p}, ..., \bar{p})$ is strictly robust to strategic uncertainty. No other strategy profile of G is robust to strategic uncertainty.*

Proof: Let $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset F$. Consider any sequence $\langle t_k \rangle_{k=1}^{\infty} \to 0$, where each $t_k > 0$. For each $k \in N$, let p^k be a Nash equilibrium of $\bar{G}^{t_k}$. Since all games $\bar{G}^{t_k}$ have the same strategy space, $[\bar{p}, p^{mon}]^n$, and this is non-empty and compact, the sequence $\langle \mathbf{p}^{t_k} \rangle_{k=1}^{\infty}$ contains a convergent subsequence with limit in $[\bar{p}, p^{mon}]^n$, according to the Bolzano-Weierstrass Theorem. Hence, without loss of generality we may assume that $\lim_{k \to \infty} p^k = p^* \in [\bar{p}, p^{mon}]^n$.

First, we prove that $p_i^* = p_j^*$ for all $i, j \in N$. For this purpose, note that $\bar{p} < p_i^k < p^{mon}$ for all i and k, and, moreover (from the proof of Proposition 3),

$$t_k v_1'\left(p_i^k\right) = v_1\left(p_i^k\right) \sum_{j \neq i} h_{ij}\left(\frac{p_i^k - p_j^k}{t_k}\right) \quad \forall i, k \tag{8}$$

where h_{ij} is the hazard-rate function of Φ_{ij}. Consider a firm $i \in N$. Suppose that $p_j^* < p_i^*$ for some $j \neq i$, and let $\varepsilon = p_i^* - p_j^* > 0$. Then, there is a K such that $p_i^k - p_j^k > \varepsilon/2$ for all $k > K$. The hazard rate being non-decreasing, we thus have

$$h_{ij}\left(\frac{p_i^k - p_j^k}{t_k}\right) \geq h_{ij}\left(\frac{\varepsilon}{2t_k}\right)$$

for that $j \neq i$ and all $k > K$. Let $\delta = h_{ij}\left[\varepsilon / \left(2t_K\right)\right] > 0$. Then

$$h_{ij}\left(\frac{p_i^k - p_j^k}{t_k}\right) \geq \delta$$

for that $j \neq i$ and all $k > K$, and hence, since all hazard rates are positive:

$$t_k v_1'\left(p_i^k\right) > \delta \cdot v_1\left(p_i^k\right)$$

for all $k > K$. However, $t_k v_1'\left(p_i^k\right) \to 0$ and $v_1\left(p_i^k\right) \to v_1\left(p_i^*\right)$ as $k \to \infty$, since v_1 is continuous, so $v_1\left(p_i^*\right) = 0$. Hence, $p_i^* = \bar{p}$. But this contradicts the hypothesis $p_i^* > p_j^* \in [\bar{p}, p^{mon}]$. Hence, $p_j^* \geq p_i^*$. Since holds for all i and $j \neq i$, we conclude that $p_j^* = p_i^*$ for all $i, j \in N$.

Secondly, we prove $p_i^* = \bar{p}$ for all $i \in N$. Since $v_1\left(p_i^k\right) > 0$ on $(\bar{p}, p^{mon})$ and all hazard rates are positive, by (8),

$$v_1\left(p_i^k\right) \cdot h_{ij}\left(\frac{p_i^k - p_j^k}{t_k}\right) \to 0 \quad \forall i, j \neq i$$

as $k \to +\infty$. Suppose that $p_i^* > \bar{p}$. Then $v_1\left(p_i^*\right) > 0$ and thus

$$h_{ij}\left(\frac{p_i^k - p_j^k}{t_k}\right) \to 0 \quad \forall j \neq i$$

implying that $p_i^k < p_j^k$ for all k sufficiently large. But, by the same token: since $p_j^* = p_i^*$, for all $j \neq i$, we also have $p_j^* > \bar{p}$ and $v_1\left(p_j^*\right) > 0$ and thus

$$h_{ji}\left(\frac{p_j^k - p_i^k}{t_k}\right) \to 0$$

implying that $p_j^k < p_i^k$ for all k sufficiently large. Both strict inequalities cannot hold. Hence, $p_i^* = \bar{p}$ for all $i \in N$. In sum: the only strategy profile that is robust to strategic uncertainty is $(\bar{p}, ..., \bar{p})$. The strict robustness claim follows immediately from the fact that the collection $\{\Phi_{ij} : \forall i \in N, j \neq i\} \subset \mathcal{F}$ above was arbitrary. **End of proof**.

Example 4. *Consider again a duopoly with identical firms, with quadratic cost function,* $C(q) = 0.2q^2$, *and linear aggregate demand:* $D(p) = \max\{0, 1 - p\}$. *Suppose that both firms' uncertainty takes the form of normally distributed noise,*

$\varepsilon_1, \varepsilon_2 \sim N(0,1)$. *We then have* $\bar{p} = 1/6 \approx 0.1667$. *The necessary first-order condition for interior* t*-equilibrium consists of the equations*

$$tv_1'(p_1) = v_1(p_1)\, h\left(\frac{p_1 - p_2}{t}\right)$$

and

$$tv_1'(p_2) = v_1(p_2)\, h\left(\frac{p_2 - p_1}{t}\right).$$

The diagram below shows these "best-reply curves" (solid and dashed, respectively) and $t = 0.1$, *with* $\bar{p}$ *marked by thin straight lines.The next diagram illustrates the*

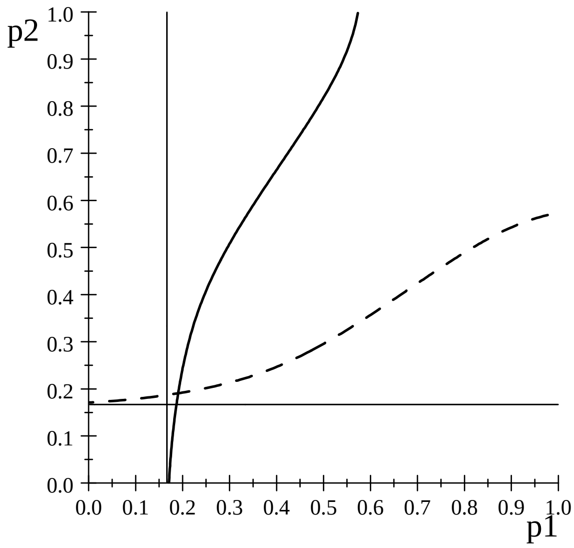

Figure 4: The best-reply curves in the perturbed pricing game.

convergence of t*-equilibria towards* $(\bar{p}, \bar{p}) = (1/6, 1/6)$. *It displays the best-reply curves of both players for* $t = 0.25$ *(solid curves),* $t = 0.1$ *(thin curves), and* $t = 0.05$ *(dashed curves). As* t *decreases, the intersection of the associated pair of curves approaches* $(\bar{p}, \bar{p})$, *the intersection between the thin horizontal and vertical lines.*

5. Conclusion

In this paper, we have investigated Bertrand games with convex costs. This is a class of games with discontinuous payoff functions and with a whole continuum of Nash equilibria Arguably, strategic uncertainty will be considerable, due to the richness of the strategy spaces and the large number of equilibria We have introduced a general

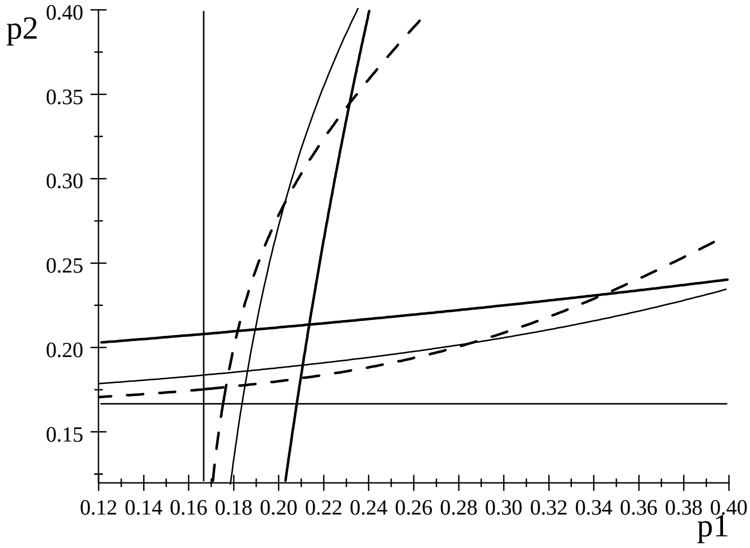

Figure 5: Convergence of t-equilibria towards $(\bar{p}, \bar{p})$

notion of robustness to strategic uncertainty and shown that this notion is powerful enough to reduce the equilibrium set to a singleton in these price-competition games. Although we here focus on a particular class of games, we believe that our concept of robustness to strategic uncertainty has a wide domain of application. Indeed, in the Nash demand game (Nash, 1953), which also has a continuum of Nash equilibria, we can show that, under symmetric strategic uncertainty, the Nash bargaining solution is the unique equilibrium outcome that is robust to strategic uncertainty.

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